

# Geometric quantization on a coset space $G/H$

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## Abstract

Geometric quantization on a coset space  $G/H$  is considered, intending to recover Mackey's inequivalent quantizations. It is found that the inequivalent quantizations can be obtained by adopting the symplectic 2-form which leads to Wong's equation. The irreducible representations of  $H$  which label the inequivalent quantizations arise from Weil's theorem, which ensures a Hermitian bundle over  $G/H$  to exist.

# 1 Introduction

Geometric quantization[19, 3, 4, 21] is the method of constructing a wave function on the phase space of a classical system and restricting it to be covariantly constant along a generalized momentum. It is a powerful approach to quantizing on a manifold which is topologically nontrivial.

In this paper, we consider quantization of a classical system on a coset space  $G/H$ , where  $G$  is a compact Lie group and  $H$  a semisimple subgroup of  $G$ . Geometric quantization has previously been applied to coset spaces interpreted as phase spaces[20]. In this study, however, a coset space  $G/H$  is regarded as a configuration space rather than a phase space, which means that the phase space of interest is the cotangent bundle  $T^*(G/H)$ .

There are many approaches to quantization on a coset space, for example, a system of imprimitivity (Mackey [1]), a canonical group (Isham [16]), a  $C^*$ -algebra (Landsman [17][2]), generalized Dirac quantization (McMullan, Tsutsui [5]) and so forth. One of the most important characteristics of quantization on  $G/H$  is the existence of inequivalent quantizations, which were found by Mackey[1]. The inequivalent quantizations on  $G/H$  are labeled by the unitary representations of  $H$ , and Hilbert spaces obtained using those quantizations belong to different sectors, if they do not belong to the same representation space of  $H$ .

The purpose of the present paper is to investigate how Mackey's inequivalent quantizations are recovered in the context of geometric quantization. We construct a symplectic 2-form on  $T^*(G/H)$  by Hamiltonian reduction from a system on  $T^*G$ . The Lagrangian leading to Wong's equation [9, 10] is used as a guiding principle to find the constraints which implement the reduction, on the ground that it allows us to identify  $H$  with the internal symmetry of the system and it is also an effective Lagrangian obtained by Mackey's quantization [5]. After that, we construct a Hilbert space on  $G/H$  and find that Weil's condition, which ensures the Hilbert space to exist, leads precisely to Mackey's inequivalent quantizations. Recently, geometric quantization of cotangent bundle with symmetry is considered in [7, 8]. Whereas these papers put an emphasis on the inequivalent quantizations of operators, our primary concern in this paper will be with the Hilbert spaces pertinent to the quantizations.

This paper is organized as follows. Section 2 is devoted to a brief review of geometric quantization. In section 3, we consider geometric quantization on a phase space whose configuration space is a coset space  $G/H$  with (and without)  $H$  as the internal symmetry. The last section is devoted to the conclusion.

## 2 A brief review of geometric quantization

We first review the method of geometric quantization which we apply later to  $G/H$ . Geometric quantization consists of two procedures, prequantization and a choice of polarization. This review is mainly based on [3, 4].

Let  $M$  be a manifold with a closed 2-form  $\omega$ . If  $\omega$  is nondegenerate, it is called a

symplectic 2-form, and we can always construct the Poisson bracket on the phase space from  $\omega$ . Since  $\omega$  is closed, we can write locally,

$$\omega = d\theta, \quad (1)$$

where  $\theta$  is a 1-form called the canonical 1-form. If  $\omega$  is degenerate,  $M$  with  $\omega$  is called the presymplectic manifold.

For simplicity, we illustrate this approach of quantization with the special case where  $M$  is a cotangent bundle of  $Q$  and  $\omega$  is symplectic.  $M$  is locally covered by the coordinate system  $(q^i, p_j)$  and  $\theta$  is locally written as  $p_i dq^i$ . We construct a Hilbert space on the classical phase space  $M$ . Consider a complex line bundle as a candidate of the Hilbert space. A wave function is defined as a section. The complex line bundle which we are interested in has following two additional structures

$$\text{connection} : \nabla\phi = (d - \frac{i}{\hbar}\theta)\phi \quad (2)$$

$$\text{Hermitian metric} : (\phi_1, \phi_2) = \phi_1^* \cdot \phi_2 \quad (3)$$

for each section  $\phi_1$  and  $\phi_2$ .  $\nabla$  denotes an inner derivative and  $\phi^*$  is the complex conjugation of  $\phi$ . We call this line bundle Hermitian line bundle and identify it with the Hilbert space on  $M$ . This procedure is prequantization.

Weil's theorem tell us that such sections on the Hermitian line bundle exist if the symplectic 2-form  $\omega$  satisfies the condition

$$\frac{1}{2\pi\hbar} \int_S \omega \in \mathbf{Z}, \quad (4)$$

where  $S$  is a 2-dimensional closed surface on  $M$ .

A physical wave function, however, must be (covariantly) constant along momenta in the position representation, in order to make the representation of operators irreducible. Therefore we impose the restriction on wave functions that the connection of the physical wave functions with respect to vectors along (generalized) momenta must vanish. The set of the vectors is called polarization, and this procedure is a choice of polarization.

In order to make an inner product invariant under a coordinate transformation (in the position representation), we introduce a 1/2-P-form  $v_q$ , which is covariantly constant along momenta, and transforms as  $v_q \mapsto (\det m)^{-\frac{1}{2}} v_q$  under the transformation that  $dq^i \mapsto m^i_j dq^j$ . Then we can define a wave function, an inner product and an operator corresponding to a classical observable  $\varrho$  as

$$\psi(q) = \phi(q) \cdot v_q \quad (5)$$

$$\langle \psi_1, \psi_2 \rangle = \int_Q \phi_1^*(q) \phi_2(q) v_q^* v_q d^n q \quad (6)$$

$$\delta_\varrho \psi = (\frac{\hbar}{i} \nabla_{\xi_\varrho} + \varrho) \phi \cdot v_q + \phi \cdot (\frac{\hbar}{i} \mathcal{L}_{\xi_\varrho}) v_q, \quad (7)$$

where  $\xi_\varrho$  is a Hamiltonian vector field defined as

$$d\varrho = -\xi_\varrho \lrcorner \omega, \quad (8)$$

and  $\mathcal{L}_{\xi_\varrho}$  is a Lie derivative which acts on  $v_q$ .

If  $\omega$  is degenerated,  $M$  is reduced to the space where  $\omega$  is symplectic. On  $M$ , there are vectors  $\{\zeta\}$  which satisfies  $\zeta \lrcorner \omega = 0$ , and the reduced space is realized by identifying points along the integral curves of  $\{\zeta\}$ . Thus if  $S^2$  in eq.(4) is a closed surface in the reduced space, then in  $M$  it is a surface  $W$  whose boundary  $\gamma$  is identified in the reduced space. This replaces eq.(4) with the condition

$$\frac{1}{2\pi\hbar} \int_W \omega = \frac{1}{2\pi\hbar} \oint_\gamma \theta \in \mathbf{Z}, \quad (9)$$

where we used Stokes' theorem.

### 3 Geometric quantization on a coset space $G/H$

In this section, we consider geometric quantization on a coset space  $G/H$ . A system we are interested in is a phase space  $T^*(G/H)$  with an internal symmetry  $H$ , which is obtained by reduction of  $T^*G$  under certain constraints.

We first show in subsection 3.1 that the naive set of constraints that “momenta” associated with  $H$  should be zero does not lead to inequivalent quantizations. Thus we impose another set of constraints which we find by comparing the Lagrangian of a free particle on  $G$  with the Lagrangian for Wong's equation, in subsection 3.2, and find that this leads to inequivalent quantizations in subsection 3.3.

#### 3.1 Quantization under naive constraint conditions

Firstly, we begin by reducing  $T^*G$  to  $T^*(G/H)$  under the naive constraints fixing momentum associated with  $H$  to be zero.

The canonical 1-form on  $T^*G$  is already given by

$$\Theta_G = \text{tr}(\tilde{R}g^{-1}dg), \quad \tilde{R} \in \mathfrak{g}, \quad (10)$$

where  $\mathfrak{g}$  is the Lie algebra of  $G$ .  $g^{-1}dg$  is a Maurer-Cartan 1-form[15], and  $\tilde{R}$  is its conjugate momentum. The Lie algebra of  $G$  can be decomposed as  $\mathfrak{g} = \{T^a\} \oplus \{T^i\}$ .  $\{T^i\}$  is a basis of the Lie algebra  $\mathfrak{h}$  of a Lie group  $H$ .  $\{T^a\}$  is a basis for  $\mathfrak{r}$  being orthogonal to  $\mathfrak{h}$ . The normalization of trace of these bases is  $\text{tr}(T^m T^n) = \delta^{mn}$ , where  $T^m, T^n \in \mathfrak{g}$ .  $\tilde{R}$  can be decomposed as

$$\tilde{R} = \tilde{R}^a T^a + \tilde{R}^i T^i.$$

In order to reduce the phase space from  $T^*G$  to  $T^*(G/H)$ , we must restrict the degrees of freedom corresponding to  $\mathfrak{h}$ . Thus we do this by imposing the constraints as follows

$$\tilde{R}^i \equiv 0, \quad (11)$$

for all  $i$ . This means that  $\tilde{R} \in \mathfrak{r}$ .

$\varpi_0 : G \rightarrow G/H$  is regarded as a bundle on  $G/H$ . Let  $\{U_\alpha\}$  be a set of open coverings of  $G/H$ . Thus  $g \in G$  in eq.(10) can be written as

$$g = \sigma_\alpha(q)h, \quad (12)$$

where  $q = \varpi_0(g) \in G/H$ ,  $h \in H$ , and  $\sigma_\alpha$  is a section such that  $\sigma_\alpha : U_\alpha \rightarrow G$ . Using these  $q$  and  $h$ , we can rewrite eq.(10) in the form

$$\Theta_G = \text{tr}(\tilde{R}h^{-1}dh) + \text{tr}(\tilde{R}h^{-1}(\sigma_\alpha^{-1}d\sigma_\alpha)h). \quad (13)$$

Since  $\tilde{R} \in \mathfrak{r}$ , the first term on the right hand side vanishes. Define as  $R = h\tilde{R}h^{-1} \in \mathfrak{r}$ , and

$$\Theta = \Theta_G|_{\tilde{R} \in \mathfrak{r}} = \text{tr}(R(\sigma_\alpha^{-1}d\sigma_\alpha)|_{\mathfrak{r}}), \quad (14)$$

where the sign  $|_{\mathfrak{r}}$  denotes the restriction to  $\mathfrak{r}$ . The symplectic 2-form  $\omega$  is defined as

$$\begin{aligned} \omega &= d\Theta \\ &= \text{tr}(dR \wedge (\sigma_\alpha^{-1}d\sigma_\alpha)|_{\mathfrak{r}}) + \text{tr}(Rd(\sigma_\alpha^{-1}d\sigma_\alpha)|_{\mathfrak{r}}). \end{aligned} \quad (15)$$

In fact, this symplectic 2-form is exact, since the canonical 1-form  $\Theta$  is defined globally. This can be seen from that  $\Theta$  is invariant under the gauge transformation  $\sigma_\alpha = \sigma_\beta h_{\beta\alpha}$  on  $U_\alpha \cap U_\beta$ , where  $(\sigma_\alpha^{-1}d\sigma_\alpha)|_{\mathfrak{r}}$  and  $R$  transforms as  $(\sigma_\alpha^{-1}d\sigma_\alpha)|_{\mathfrak{r}} \rightarrow (h_{\beta\alpha}^{-1}\sigma_\beta^{-1}d\sigma_\beta h_{\beta\alpha})|_{\mathfrak{r}}$  and  $R \rightarrow (h_{\beta\alpha}^{-1}Rh_{\beta\alpha})|_{\mathfrak{r}}$ .

Secondly, we consider prequantization. The quantization condition on  $T^*(G/H)$  is satisfied, because

$$\frac{1}{2\pi\hbar} \int_S \omega = \frac{1}{2\pi\hbar} \int_S d\Theta = 0 \in \mathbf{Z}, \quad (16)$$

where  $S$  is a 2-dimensional closed surface on  $T^*(G/H)$ .

Thus we can define a Hermitian line bundle  $L$  on  $T^*(G/H)$ . Let  $\Gamma(L)$  denote a set of sections on  $L$ . The connection for  $\Gamma(L)$  is defined as

$$\nabla\phi = d\phi - \frac{i}{\hbar}\Theta\phi, \quad (17)$$

for  $\phi(R, q) \in \Gamma(L)$ . The Hermitian metric compatible with this connection is

$$(\phi_1, \phi_2)(R, q) = \phi_1^*(R, q)\phi_2(R, q), \quad (18)$$

where  $\phi_1, \phi_2 \in \Gamma(L)$ .

Lastly, we choose a polarization as  $\{\frac{\partial}{\partial R_a}\}$ . Thus

$$\nabla_{\frac{\partial}{\partial R_a}}\phi = \frac{\partial}{\partial R_a}\phi = 0. \quad (19)$$

Therefore,  $\phi$  depends only on  $q$ , and can be written as  $\phi(q)$ . Let  $v_q(\{\frac{\partial}{\partial R_a}\})$  denotes the 1/2-P-form. The wave function is written as  $\phi(q)v_q$ . The inner product of wave functions is defined as  $\int_{G/H}(\phi_1, \phi_2)(q)v_q^*v_q d^n q$ .

Inequivalent quantizations do not emerge under the constraint eq.(11), as can be seen from eq.(16). This is a situation similar to the case of Dirac's naive constraints in [5]. Now we discuss what constraints should be imposed on  $T^*G$  to obtain symplectic 2-form on  $T^*(G/H)$  which correspond to inequivalent quantizations.

### 3.2 Derivation of the symplectic 2-form on $G/H$ leading to inequivalent quantizations

We start by decomposing the degrees of freedom of a system on  $G$  to  $G/H$  and  $H$ , and investigate what constraints should be imposed.

The first order Lagrangian of a free particle on  $G$  is

$$\mathbf{L} = \text{tr}(\tilde{R}g^{-1}\dot{g}) - \frac{1}{2}\text{tr}(\tilde{R}^2). \quad (20)$$

One can see that this lead to the Lagrangian  $\mathbf{L} = \frac{1}{2}\text{tr}(g^{-1}\dot{g})^2$  by eliminating  $\tilde{R}$ , using equations of motion for  $\tilde{R}$ .

We rewrite  $g^{-1}dg$ , using eq.(12), as

$$\begin{aligned} g^{-1}dg &= h^{-1}\sigma_\alpha^{-1}d\sigma_\alpha h + h^{-1}dh \\ &= (h^{-1}\sigma_\alpha^{-1}d\sigma_\alpha h + h^{-1}dh)|_{\mathfrak{h}} + (h^{-1}\sigma_\alpha^{-1}d\sigma_\alpha h)|_{\mathfrak{r}}, \end{aligned} \quad (21)$$

where  $|_{\mathfrak{h}}$  denotes the restriction to  $\mathfrak{h}$ . Here we define  $A = (\sigma_\alpha^{-1}d\sigma_\alpha)|_{\mathfrak{h}}$ , called H-connection[2, 5], because, under the gauge transformation as  $\sigma_\alpha \mapsto \sigma_\alpha h$ ,  $(\sigma_\alpha^{-1}d\sigma_\alpha)|_{\mathfrak{h}}$  transforms as a vector potential of non Abelian gauge field whose gauge group is  $H$ . Using eq.(21), we can rewrite eq.(20) as

$$\begin{aligned} \mathbf{L} &= \text{tr}(\tilde{R}|_{\mathfrak{r}}(h^{-1}\sigma_\alpha^{-1}\dot{\sigma}_\alpha h)|_{\mathfrak{r}}) - \frac{1}{2}\text{tr}((\tilde{R}|_{\mathfrak{r}})^2) \\ &\quad + \text{tr}(\tilde{R}|_{\mathfrak{h}}h^{-1}\dot{h}) + \text{tr}(h\tilde{R}|_{\mathfrak{h}}h^{-1}\dot{A}) - \frac{1}{2}\text{tr}((\tilde{R}|_{\mathfrak{h}})^2), \end{aligned} \quad (22)$$

where we use  $\dot{A} = A_a \dot{q}^a = (\sigma_\alpha^{-1}\dot{\sigma}_\alpha)|_{\mathfrak{h}}$ .

By using the equations of motion for  $\tilde{R}_a$

$$\tilde{R}_a = (h^{-1}\sigma_\alpha^{-1}\dot{\sigma}_\alpha h)_a, \quad (23)$$

we substitute this for  $\tilde{R}_a$  in eq.(22), and eliminate  $\tilde{R}_a$ . Thus the Lagrangian (22) becomes

$$\mathbf{L} = \frac{1}{2}\text{tr}((\sigma_\alpha^{-1}\dot{\sigma}_\alpha|_{\mathfrak{r}})^2) + \text{tr}(\tilde{R}|_{\mathfrak{h}}h^{-1}\dot{h}) + \text{tr}(h\tilde{R}|_{\mathfrak{h}}h^{-1}\dot{A}) - \frac{1}{2}\text{tr}((\tilde{R}|_{\mathfrak{h}})^2), \quad (24)$$

where  $\sigma_\alpha^{-1}\dot{\sigma}_\alpha|_{\mathfrak{r}} = \sigma_\alpha^{-1}\partial_a\sigma_\alpha|_{\mathfrak{r}}\dot{q}^a$ , and the metric on  $G/H$  is given by  $g_{ab} = \text{tr}(\sigma^{-1}\partial_a\sigma|_{\mathfrak{r}}\sigma^{-1}\partial_b\sigma|_{\mathfrak{r}})$ .

Now we recall Wong's equation is known to be an effective lagrangian which leads to Mackey's inequivalent quantizations[5] and it describes a system coupling with non Abelian gauge field[9, 10]. Here we make use of the Lagrangian for Wong's equation to identify  $H$  as the gauge group. The Lagrangian is

$$\mathbf{L} = \frac{1}{2}g_{ab}\dot{q}^a\dot{q}^b + i\hbar\text{tr}(Kh^{-1}\dot{h}) + i\hbar\text{tr}(hKh^{-1}\dot{A}), \quad (25)$$

where  $K$  is a constant element in  $\mathfrak{h}$ .

Obviously, eq.(24) and eq.(25) have the same form except for  $\tilde{R}|_{\mathfrak{h}}$  and  $K$  [5]. By identifying these two equations, therefore, it is possible to recognize that eq.(24) describes quantum theory whose classical configuration space is  $G/H$ , and whose internal symmetry is  $H$ . Consequently, a constraint which we should impose is

$$\tilde{R}|_{\mathfrak{h}} \equiv i\hbar K. \quad (26)$$

The basis of  $\mathfrak{h}$  is defined so that its Cartan subalgebra commutes with  $K$ . We neglect the term  $-\frac{1}{2}\text{tr}((\tilde{R}|_{\mathfrak{h}})^2)$  in eq.(24), because it is constant under the constraints, eq.(26).

Under the constraints, the canonical 1-form on  $T^*G$  in eq.(10) is reduced on  $T^*(G/H)$  to

$$\Theta = R_a(\sigma_\alpha^{-1}d\sigma_\alpha)^a + i\hbar\text{tr}(hKh^{-1}A) + i\hbar\text{tr}(Kh^{-1}dh). \quad (27)$$

The closed 2-form is

$$\begin{aligned} \Omega &= d\Theta \\ &= dR_a \wedge (\sigma_\alpha^{-1}d\sigma_\alpha)^a + R_a d(\sigma_\alpha^{-1}d\sigma_\alpha)^a + i\hbar\text{tr}(d(hKh^{-1}) \wedge A) \\ &\quad + i\hbar\text{tr}(hKh^{-1}dA) - i\hbar\text{tr}(Kh^{-1}dh \wedge h^{-1}dh). \end{aligned} \quad (28)$$

Let  $\Delta$  be a set of roots of  $\mathfrak{h}$ .  $\beta_1, \beta_2 \dots \in \Delta$  denote roots of  $\mathfrak{h}$ . With respect to the roots, we can decompose (complexified)  $\mathfrak{h}$  as

$$\mathfrak{h} = \mathcal{T} \oplus \mathfrak{h}_{\beta_1} \oplus \mathfrak{h}_{\beta_2} \oplus \dots, \quad (29)$$

where  $\mathcal{T}$  is a space spanned by the Cartan subalgebra of  $\mathfrak{h}$ , and  $\mathfrak{h}_{\beta_i}$  is an eigenspace of  $\mathcal{T}$  and a space whose elements has  $\beta_i$  as a root[5, 13]. Let  $E_{\beta_i} \in \mathfrak{h}_{\beta_i}$ ,  $E_{\beta_j} \in \mathfrak{h}_{\beta_j}$ , then

$$[E_{\beta_i}, E_{\beta_j}] \begin{cases} \in \mathfrak{h}_{\beta_i+\beta_j} & \text{if } \beta_i + \beta_j \in \Delta, \\ = \frac{2(\beta_i)_k}{|\beta_i|^2} H_k = H_{\beta_i} & \text{if } \beta_i = -\beta_j, \\ = 0 & \text{otherwise,} \end{cases} \quad (30)$$

where  $\{H_i\}$  is an orthonormal basis of  $\mathcal{T}$ . Since  $h^{-1}dh \in \mathfrak{h}$ ,  $h^{-1}dh$  can be expanded as

$$h^{-1}dh = \sum_{\alpha_j \in \Delta_s} b_j H_{\alpha_j} + \sum_{\varphi_j \in \Delta^+} (B_{\varphi_j} E_{\varphi_j} + B_{-\varphi_j} E_{-\varphi_j}), \quad (31)$$

where  $\Delta_s$  is a set of simple roots and  $\Delta^+$  is a set of positive roots. And  $B_{-\varphi_j} = -B_{\varphi_j}^*$ , because  $h^{-1}dh$  is anti-Hermite and  $E_{\varphi_i}^\dagger = E_{-\varphi_i}$ . ( $B^*$  denotes the complex conjugation of  $B$ .) We substitute this for  $h^{-1}dh$  in eq.(28). Since  $K = K^i H_i \in \mathcal{T}$ , we find

$$\begin{aligned} &\text{tr}(Kh^{-1}dh \wedge h^{-1}dh) \\ &= \text{tr}(K(b_i H_{\alpha_i} + B_{\varphi_i} E_{\varphi_i} + B_{-\varphi_i} E_{-\varphi_i}) \wedge (b_j H_{\alpha_j} + B_{\varphi_j} E_{\varphi_j} + B_{-\varphi_j} E_{-\varphi_j})) \\ &= \text{tr}(K(\frac{1}{2}[H_{\alpha_i}, H_{\alpha_k}] b_i \wedge b_k + [H_{\alpha_i}, E_{\varphi_k}] b_i \wedge B_{\varphi_k} \\ &\quad + [H_{\alpha_i}, E_{-\varphi_k}] b_i \wedge B_{-\varphi_k} + \frac{1}{2}[E_{\varphi_k}, E_{\varphi_l}] B_{\varphi_k} \wedge B_{\varphi_l} \\ &\quad + \frac{1}{2}[E_{-\varphi_k}, E_{-\varphi_l}] B_{-\varphi_k} \wedge B_{-\varphi_l} + [E_{\varphi_k}, E_{-\varphi_l}] B_{\varphi_k} \wedge B_{-\varphi_l})) \\ &= - \sum_{\varphi_j \in \Delta^+} K_{\varphi_j} B_{\varphi_j} \wedge B_{\varphi_j}^*, \end{aligned} \quad (32)$$

where  $K_{\varphi_j} = \text{tr}(KH_{\varphi_j}) = \frac{2K^i(\varphi_j)_i}{|\varphi_j|^2}$ . Thus, the closed 2-form is

$$\begin{aligned}\Omega &= dR_a \wedge (\sigma_a^{-1} d\sigma_a)^a + R_a d(\sigma_a^{-1} d\sigma_a)^a + i\hbar \text{tr}(d(hKh^{-1}) \wedge A) \\ &\quad + i\hbar \text{tr}(hKh^{-1} dA) + i\hbar \sum_{\varphi_j \in \Delta^+} K_{\varphi_j} B_{\varphi_j} \wedge B_{\varphi_j}^*.\end{aligned}\quad (33)$$

In fact, this 2-form is still not symplectic because it is degenerated. To see that, we first define duals of  $b_i$  and  $B_{\varphi_j}$  as  $y_j$  and  $Y_{\varphi_j}$ , respectively,  $\langle b_i, y_j \rangle = \delta_{ij}$ ,  $\langle B_{\varphi_i}, Y_{\varphi_j} \rangle = \delta_{\varphi_i, \varphi_j}$ . In particular, we examine the term  $d(hKh^{-1})$  in eq.(33), and it can be rewrote as

$$\begin{aligned}d(hKh^{-1}) &= dhKh^{-1} - hKh^{-1}dh h^{-1} \\ &= h(h^{-1}dh)Kh^{-1} - hK(h^{-1}dh)h^{-1}.\end{aligned}\quad (34)$$

And we can immediately find that the Cartan subalgebra component of  $h^{-1}dh$  vanishes in eq.(34), because

$$h(b_i H_{\alpha_i})Kh^{-1} - hK(b_i H_{\alpha_i})h^{-1} = 0, \quad (35)$$

where we use that  $b_i H_{\alpha_i}$  and  $K$  commute. Hence,  $y_i]d(h^{-1}Kh) = 0$ , and thus,  $y_j]\Omega = 0$ . In contrast,  $\frac{\partial}{\partial R_a}] \Omega$ ,  $Z_a]\Omega$ ,  $Y_{\varphi_i}] \Omega$ ,  $Y_{\varphi_i}^*]\Omega$  are nonzero, where  $Z_b$  is defined by  $\langle (\sigma_a^{-1} d\sigma_a)_a, Z_b \rangle = \delta_{ab}$ . Thus the condition that a Hermitian line bundle exists is not eq.(4) but eq.(9).

Let  $\mathcal{K}_y$  be an integral manifold of  $\{y_i\}[4]$ , which is a manifold filled with integral curves induced by  $\{y_i\}$ . In order to define a well-defined Hermitian line bundle on  $T^*(G/H)$  with internal symmetry  $H$ , the condition

$$\frac{1}{2\pi\hbar} \oint_{\gamma} \Theta = \frac{1}{2\pi\hbar} \oint_{\gamma} \text{tr}(i\hbar Kh^{-1} dh) \in \mathbf{Z}. \quad (36)$$

must hold for all one-dimensional closed curves  $\gamma \subset \mathcal{K}_y$ , as we mentioned in section 2. Since  $h$  on  $\gamma \subset \mathcal{K}_y$  is parametrized as  $e^{-i\theta^i(t)H_{\alpha_i}}$  where  $\theta^i(T) - \theta^i(0) = 2\pi n^i$  ( $n^i \in \mathbf{Z}$ ), we find

$$\frac{1}{2\pi\hbar} \oint_{\gamma} \Theta = \frac{1}{2\pi} \int_{\theta^i(0)}^{\theta^i(T)} K_{\alpha_i} d\theta^i = K_{\alpha_i} n^i. \quad (37)$$

Thus<sup>1</sup>, for  $\forall i = 1, 2, \dots, \dim H$ ,

$$K_{\alpha_i} \in \mathbf{Z}. \quad (38)$$

The  $K_{\alpha_i}$ 's label the inequivalent quantum theories of this system.

Here we notice that the condition for Mackey's inequivalent quantizations [5] is obtained by Weil's condition, eq.(36), in the context of geometric quantization.

### 3.3 Quantization on $G/H$

We have obtained the symplectic 2-form which corresponds to Mackey's inequivalent quantizations, and let us apply geometric quantization procedure to this system.

We start with prequantization. Let  $S_K$  be the subgroup of  $H$  which leaves  $K$  invariant under the coadjoint action such that  $s^{-1}Ks = K$  for all  $s \in S_K$ .  $S_K$  can be identified with  $\mathcal{K}_y$ , because an element of  $S_K$  is obtained by the exponential map of elements of

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<sup>1</sup>This is suggested in [7].



Cartan subalgebra  $\mathcal{T}$ . The symplectic 2-form  $\Omega$  defined in eq.(33) is invariant under gauge transformation associated with  $S_K$ . Therefore,  $T^*(G/H) \times (H/S_K)$  is the classical phase space to be quantized.

If  $K$  satisfies the condition of eq.(38), we can define a line bundle  $\varpi : L \rightarrow T^*(G/H) \times (H/S_K)$ . Let  $\phi^{(K)}(R, q, h)$  be a section on  $L$ . From the above argument,  $\phi^{(K)}(R, q, h)$  should be invariant under the right action of  $s \in S_K$  up to a phase factor. In other words,  $\phi^{(K)}(R, q, h)$  should be constant under the parallel translation in the direction of  $\{y_i\}$ , because we identify all points in  $\mathcal{K}_y$ . The Hermitian metric and connection for sections are given as

$$(\phi_1^{(K)}, \phi_2^{(K)})(R, q, h) = \phi_1^{(K)*}(R, q, h) \phi_2^{(K)}(R, q, h) \quad (39)$$

$$\nabla \phi^{(K)}(R, q, h) = (d - \frac{i}{\hbar} \Theta) \phi^{(K)}(R, q, h), \quad (40)$$

respectively. Thus  $\phi^{(K)}(R, q, h)$  satisfies the condition  $\nabla_{y_i} \phi^{(K)}(R, q, h) = 0$ , and we obtain its transformation under the action of  $s = e^{-i\theta^i H_{\alpha_i}} \in S_K$  as

$$\begin{aligned} \phi^{(K)}(R, q, hs) &= \exp(-\int_e^s \text{tr}(K(h^{-1}dh))) \phi^{(K)}(R, q, h) \\ &= \exp(iK_{\alpha_i} \theta^i) \phi^{(K)}(R, q, h). \end{aligned} \quad (41)$$

Secondly, we choose a polarization.  $\Omega$  which is defined in eq.(33) shows us that it is possible to choose a polarization as the set  $\{\frac{\partial}{\partial R_a}, Y_{\varphi_i}^*\}$ . Thus, conditions which a quantizable section fulfills is that

$$\nabla_{\frac{\partial}{\partial R_a}} \phi^{(K)}(R, q, h) = 0; \quad (42)$$

$$\nabla_{Y_{\varphi_i}^*} \phi^{(K)}(R, q, h) = 0. \quad (43)$$

Eq.(42) shows that the section  $\phi^{(K)}$  does not depend on  $R_a$ 's. Thus we let  $\phi^{(K)}(q, h)$  represent  $\phi^{(K)}$  as the solution of eq.(42).

Let us consider the condition eq.(43). Though the solution of the equation is given in [5], we explain the argument done there for completeness. Recall the orthonormal condition of an irreducible unitary representation,  $\pi_{\mu\nu}^\chi(h)$ , of  $H$

$$d_\chi \int_H d\mu(h) \pi_{\mu\nu}^\chi(h) \pi_{\rho\sigma}^{\chi'*}(h) = \delta_{\mu\rho} \delta_{\nu\sigma} \delta^{\chi\chi'} V_H, \quad (44)$$

where  $\pi_{\mu\nu}^{\chi'*}$  represents a complex conjugation of  $\pi_{\mu\nu}^\chi$ .  $d_\chi$  represents the dimension of the representation whose highest weight is  $\chi$  and  $V_H$  the volume of  $H$  ( $= \int_H d\mu(h)$ ), respectively, and  $d\mu(h)$  is a Haar measure of  $H$ . Since  $\pi_{\mu\nu}^\chi(h)$  spans a complete set, one can expand the section as

$$\phi^{(K)}(q, h) = \sum_{\chi, \mu, \nu} \tilde{\phi}_{\mu\nu}^\chi(q) \pi_{\mu\nu}^{\chi*}(h). \quad (45)$$

Note that we can interpret  $\pi_{\mu\nu}^\chi(h)$  as

$$\pi_{\mu\nu}^\chi(h) = \langle \chi, \mu | \pi^\chi(h) | \chi, \nu \rangle, \quad (46)$$

where  $|\chi, \mu\rangle$  is an eigenstate of the Cartan subalgebra of  $\mathfrak{h}$ :  $H_{\beta_i}|\chi, \mu\rangle = \mu_{(\beta_i)}|\chi, \mu\rangle$ , where  $\mu_{(\beta_i)} = \frac{2\beta_i \cdot \mu}{|\beta_i|^2}$ .  $Y_{\varphi_i}$  is the dual of  $B_{\varphi_i} = \text{tr}(h^{-1}dhE_{-\varphi_i})$ . The explicit form of  $Y_{\varphi_i}$  is

$$Y_{\varphi_i} = C \cdot \text{tr}(hE_{\varphi_i} \frac{\partial}{\partial h^T}), \quad (47)$$

where  $C = \{\text{tr}(E_{\varphi_i} E_{-\varphi_i})\}^{-1}$ . Eq.(47) indicates that  $Y_{\varphi_i}$  is a left-invariant vector field. Thus,

$$\begin{aligned} Y_{\varphi_i}^* \pi_{\mu\nu}^*(h) &= C \cdot \langle \chi, \mu | \pi^\chi(hE_{\varphi_i}) | \chi, \nu \rangle^* \\ &= C \cdot \langle \chi, \mu | \pi^\chi(h) \pi^\chi(E_{\varphi_i}) | \chi, \nu \rangle^* \end{aligned} \quad (48)$$

This is equal to zero if  $\nu = \chi$ . Thus, owing to eq.(45),  $\phi^{(K)}(q, h)$  which fulfills the condition eq.(43), is

$$\phi^{(K)}(q, h) = \sum_{\chi, \mu} \tilde{\phi}_{\mu\chi}^\chi(q) \pi_{\mu\chi}^*(h). \quad (49)$$

For highest weight, we use the notation  $H_{\alpha_i}|\chi, \chi\rangle = \chi_{(\alpha_i)}|\chi, \chi\rangle$ , and  $\chi_{(\alpha_i)} \in \mathbf{Z}$ . Since the equation that  $\pi_{\mu\chi}^*(s) = \delta_{\mu\chi} \exp(i\chi_{(\alpha_i)}\theta^i)$  holds for  $s \in S_K$  defined in eq.(41),

$$\phi^{(K)}(q, hs) = \sum_{\chi, \mu} \tilde{\phi}_{\mu\chi}^\chi(q) \pi_{\mu\chi}^*(h) \exp(i\chi_{(\alpha_i)}\theta^i). \quad (50)$$

Thus, we find that the section which we want is the component of  $\chi_{(\alpha_i)} = K_{\alpha_i}$ , comparing eq.(50) with eq.(41).

A wave function on  $G/H$  with an internal symmetry  $H$  is the form that

$$\phi^{(K)}(q, h)v = \sum_{\mu} \psi_{\mu}(q) \pi_{\mu\chi}^*(h)v, \quad (51)$$

where  $v$  is a 1/2-P-form on  $G/H$ .

The inner product of the wave functions is

$$\begin{aligned} \langle \phi_1^{(K)}, \phi_2^{(K)} \rangle &= \int_{G/H} v^* v d^n q \left( \frac{d\chi}{V_H} \right) \int_H d\mu(h) (\phi_1^{(K)}, \phi_2^{(K)})(q, h) \\ &= \int_{G/H} d^n q \sum_{\mu} \psi_{1\mu}^\dagger(q) \psi_{2\mu}(q) v^* v, \end{aligned} \quad (52)$$

where we use eq.(44) and eq.(51) for the last equality.

The operator corresponding to a classical observable  $\varrho$  is

$$\hat{\delta}_\varrho = \frac{\hbar}{i} \nabla_{\xi_\varrho} + \varrho + \frac{\hbar}{i} \mathcal{L}_{\xi_\varrho}. \quad (53)$$

Note that all physical information of the wave function is contained in  $\psi_\mu(q)$  as can be easily seen from eq.(52). We can solve eq.(51) with respect to  $\psi_\mu(q)$  using eq.(44) as

$$\psi_\mu(q)v = \frac{d\chi}{V_H} \int_H d\mu(h) \phi^{(K)}(q, h) \pi_{\mu\chi}^\chi(h)v. \quad (54)$$

Thus,  $\psi_\mu(q)v$  is regarded as a physical wave function and is labeled by the character of the representations of  $H$ . This shows that our method by geometric quantization reproduces Mackey's inequivalent quantizations [1] and is consistent with other approaches [17, 16, 5] as well.

## 4 Conclusion

In this paper we have quantized a classical system on a coset space  $G/H$  based on the method of geometric quantization.

We constructed the classical system on  $T^*(G/H)$  by Hamiltonian reduction from  $T^*G$ . The naive set of constraints in eq.(11) which implements the reduction was found to lead only to the trivial sector of inequivalent quantizations. However, the comparison of the Lagrangian for a free particle on  $G$  with the one that leads to Wong's equation yields the guiding principle for finding the symplectic 2-form leading to Mackey's inequivalent quantizations.

The important point is that the inequivalent quantizations derive from Weil's theorem applied to presymplectic manifolds. In contrast to [7, 8] which characterized the superselection sectors by operators, we did so by Hilbert spaces. This characterization arises from the procedure of restricting wave functions to those which are covariantly constant along the integral curves of the polarization.

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